SYMPLECTIC POTENTIALS ASSOCIATED WITH CUTTING CONSTRUCTION OF TORIC SYMPLECTIC AND CONTACT MANIFOLDS

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Abstract. We introduce cutting construction of possibly non-compact symplectic toric manifolds and describe explicitly canonical symplectic potential on symplectic toric manifolds, in particular, symplectic cones that correspond to a weakly convex good cone. As an application we define canonical almost compact metric structure on compact connected contact toric manifolds of non-Sasakian type. We further prove there are no toric Sasakian structures on these manifolds.

1. Introduction

The history of the classification of symplectic toric manifolds began with the classification theorem of compact symplectic toric manifolds by T.Delzant [D]. Thereafter, E.Lerman showed the classification theorem of compact connected contact toric (c.c.c.t. for short) manifolds [L3], and Y.Karshon and E.Lerman showed the classification theorem of non-compact symplectic toric manifolds [KL]. The moment map on symplectic toric manifolds plays an important role in these classification theorems. Roughly speaking, there is a one-to-one correspondence between symplectic toric manifolds and certain convex sets except free action case. Hence it is important to construct symplectic toric manifolds by convex sets. After Delzant's work, such a construction is called the Delzant construction and the corresponding convex sets are called the Delzant polytopes (c.f. subsection 2.2). This construction also has complex geometric aspects and so constructed manifolds admit the canonical Kähler structure. V.Guillemin studied this canonical Kähler structure in detail and have the following symplectic potential by the Legendre transformation:
Let $\Delta = \{x \in (\mathbb{R}^n)^* \mid \langle x, \eta_i\rangle \leqq \kappa_i, i = 1, \cdots, N\}$ be a Delzant polytope and we set

$$\eta_1 = \begin{pmatrix} \eta_1^1 \\ \cdot \\ \cdot \\ \cdot \\ \eta_1^n \end{pmatrix}, \cdots, \eta_N = \begin{pmatrix} \eta_N^1 \\ \cdot \\ \cdot \\ \cdot \\ \eta_N^n \end{pmatrix}, l_1(x) := \kappa_1 - \langle x, \eta_1\rangle, \cdots, l_N(x) := \kappa_N - \langle x, \eta_N\rangle, l_\infty(x) := \sum l_i(x).$$

Then Guillemin's symplectic potential is the function $\mathrm{G}_\Delta : \mathring{\Delta} \to \mathbb{R}$;

$$\mathrm{G}_\Delta(x) := \frac{1}{2}\sum l_i(x)\log l_i(x) - \frac{1}{2} l_\infty(x)$$

where $\mathring{\Delta}$ denotes the interior of Δ. For more details on this see [A, G].

However in the classification theorem of non-compact symplectic toric manifolds, there are no canonical Kähler structures on some of the constructed symplectic toric manifolds, because the construction called collapsing (c.f. [KL] section4) does not have Kähler geometric aspects. Therefore it is reasonable to ask if there is a canonical Kähler structure on constructed symplectic toric manifolds and what it is if it exists.

In this paper, we define a new class of convex sets called unimodular sets (c.f. Definition 2.1), introduce cutting construction (c.f. Theorem 2.8) which is Kähler geometric method and describe explicitly the canonical Kähler structure and the symplectic potential on corresponding manifolds. In our situation, the symplectic potential is determined by cutting construction rather than the Legendre transformation. However we have the following symplectic potential which is similar to Guillemin's one:

Theorem 1.1. *Let $\Delta = \{x \in (\mathbb{R}^n)^* \mid \langle x, \eta_i\rangle \leqq \kappa_i, i = 1, \cdots, N\}$ be a unimodular set, and suppose $(M_\Delta, \omega_\Delta, T^n, \Phi_\Delta)$ is a symplectic toric manifold which is constructed by cutting construction. Then*

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there is a canonical Kähler structure $(\omega_\Delta, J_\Delta, g_\Delta)$ *on* M_Δ *which is given by the symplectic potential;*

$$\mathrm{Sp}(x) = \frac{1}{2}\|x\|^2 + \frac{1}{2}\sum l_i(x)\log l_i(x) - \frac{1}{2}l_\infty(x).$$

Another application of cutting construction is to construct a c.c.c.t. manifold which corresponds to weakly convex cone (c.f. Definition 2.2). These manifolds were not constructed in E.Lerman's paper [L3], witch has been pointed out in [Y]. We further prove there do not exist toric Sasakian structures on these manifolds.

This paper is organized as follows. In section 2, we introduce cutting construction and its proof. In section 3, we observe the diffeomorphism types of manifolds which correspond to weakly convex good cones. In section 4, we interpret cutting construction in term of Kähler geometry. Then we compute the canonical Kähler structure of cutting constructed manifolds and their symplectic potential explicitly. In section 5, we apply cutting construction to contact manifolds. Then we have c.c.c.t. manifolds. In particular, we have c.c.c.t. manifolds which correspond to weakly convex good cones. We further prove there do not exist toric Sasakian structures on these manifolds.

2. Preliminaries

2.1. **Basic facts and basic notations.** Let (M,ω) be a symplectic manifold with symplectic form ω and suppose G is a Lie group with Lie algebra $\mathfrak{g}$. Let $\Psi : G \to \mathrm{Sympl}(M,\omega)$ be a symplectic action of G where $\mathrm{Sympl}(M,\omega)$ is the group of symplectomorphisms that map M to itself. The action Ψ is a Hamiltonian action if there exists a map $\mu : M \to \mathfrak{g}^*$ satisfing:

(1) For each $X \in \mathfrak{g}$, let X_M be the vector field on M induced by the one-parameter subgroup $\{\exp(tX) \mid t \in \mathbb{R}\} \subset G$, then

$$\iota_{X_M}\omega = -d\langle \mu, X\rangle \tag{2.1}$$

where ι is the interior product operator, and $\langle\cdot,\cdot\rangle$ is the algebraic pairing.

(2) μ is equivariant with respect to the given action Ψ of G on M and the coadjoint action ad^* of G on $\mathfrak{g}^*$:

$$\mu \circ \Psi_g = \mathrm{ad}^*_g \circ \mu \ \ ({}^\forall g \in G\,). \tag{2.2}$$

The quadruple (M,ω,G,μ) is then called a Hamiltonian G-space and μ is the moment map.

Suppose G is commutative. Then since the coadjoint action is trivial, the above equivariance becomes invariance. For $g \in G$ and $p \in M$ we denote $\Psi_g(p)$ by $g \cdot p$.

Now we define a symplectic toric manifold to be a connected symplectic manifold (M^{2n},ω) equipped with an effective Hamiltonian action of a torus T^n and with a choice of a corresponding moment map μ.

In this paper we confine ourselves to the case where G is a torus T^n, i.e. we only deal with toric cases. Hence we identify the Lie algebra of torus $G = T^n$ with $\mathbb{R}^n$ for the sake of convenience. It is well known that a toric symplecic manifold corresponds to a certain convex polyhedral set.

Recall that convex polyhedral set in $(\mathbb{R}^n)^* = \mathfrak{g}^*$ is

$$\Pi = \{x \in (\mathbb{R}^n)^* \mid \langle x, \eta_i\rangle \leqq \kappa_i, i = 1,\cdots,N\} \tag{2.3}$$

where $\eta_i \in \mathbb{R}^n$ (this is called outward conormal vector of the i-th facet) and $\kappa_i \in \mathbb{R}$. Without loss of generality, we can assume that the set $\{\eta_i\}$ is minimal, i.e. that for any index j

$$\Pi \neq \{x \in (\mathbb{R}^n)^* \mid \langle x, \eta_i\rangle \leqq \kappa_i, i \neq j\}, \tag{2.4}$$

which we assume throughout this paper.

Definition 2.1 (unimodular set). *Let* Π *be as above. A unimodular set* Δ *is a relatively open subset of* Π *satisfying:*
(1) if Δ *has vertices then each vertex is simple, i.e. there are* n *edges meeting at each vertex;*
(2) each outward conormal vector of a facet of Δ *is in* $\mathbb{Z}^n$ *and primitive;*
(3) for any subset $I \subset \{1,\cdots,N\}$*, the following holds: if* $\Delta \cap F_I \neq \emptyset$ *then* $\{\eta_i\}_{i\in I}$ *is a basis of integral lattice of a subtorus* $K \subset T^n$*, where* $F_I := \{x \in \Pi \mid \langle x,\eta_i\rangle = \kappa_i, i \in I\}$

Definition 2.2 (strongly convex and weakly convex). *An* n*-dimensional convex polyhedral set* Π *is str-ongly convex iff* $\mathbb{R}$*-span*$\{\eta_1,\cdots,\eta_N\} = (\mathbb{R}^n)^*$*, and a unimodular set* $\Delta = {}^\exists U \cap \Pi$ *where* U *is an open subset of* $(\mathbb{R}^n)^*$ *is strongly convex iff* Π *is strongly convex. A unimodular set is weakly convex iff it is*

not strongly convex. Note that the strongly or weakly convexity of a unimodular set $\Delta = U \cap \Pi$ depend on the ambient polyhedral set Π, hence we should consider a unimodular set $\Delta = U \cap \Pi$ coincides with another one $\Delta' = U' \cap \Pi'$ iff $\Delta = \Delta'$ and $\Pi = \Pi'$.

We note that if Π is a convex polyhedral cone, this definition of strongly convexity is equivalent to the usual one.

The source of the word *unimodular* is the paper of Y.Karshon and E.Lerman([KL]). If a unimodular set $\Pi = \Delta$ is a polytope, i.e. it is a *compact* polyhedral set, then it is called a Delzant polytope, and the condition is referred to as simple, (2) to as rational, and (3) to as smooth. Notice that a Delzant polytope is necessarily strongly convex. If a unimodular set is a cone without the conical point then it is a good cone without the conical point (cf. Definition 5.15). That is the moment image of a symplectic cone. Note that if a cone that corresponds to a symplectic cone is weakly convex, we define that the conical point is the origin.

For a Delzant polytope, we have the following well-known result by T.Delzant [D]:

Theorem 2.3 (Delzant)**.** *Compact connected symplectic toric manifolds are classified by Delzant polytopes. More precisely, there is a one-to-one correspondence between n-dimensional Delzant polytopes and $2n$-dimensional compact symplectic manifolds up to T^n-equivariant symplectomorphisms that preserve a moment map.*

Remark 2.4. *This theorem is generalized to a non-compact and disconnected case by Y.Karshon and E.Lerman* [KL]. *Their result says there is a similar correspondence between unimodular sets and symplectic toric manifolds: For an n-dimensional unimodular set Δ which satisfies $H^2(\Delta; \mathbb{Z}^n \times \mathbb{R}) = 0$, there corresponds a symplectic toric manifold $(M_\Delta, \omega_\Delta, T^n, \mu_\Delta)$ whose moment image is Δ. Moreover such an M_Δ is determined by Δ up to T^n-equivariant symplectomorphisms that preserve a moment map. We will give a construction of such M_Δ's bellow.*

In general, the moment image of a symplectic toric manifold is not a unimodular set and symplectic toric manidfolds can not be classified by its moment images (*see* [KL]).

We recall the Denlzant construction which is in common use to prove the existence part in Theorem 2.3.

2.2. **Delzant construction.** Suppose we are given a Delzant polytope $\Delta = \{x \in (\mathbb{R}^n)^* \mid \langle x, \eta_i \rangle \leqq \kappa_i, i = 1, \cdots, N\}$. Let $\{e_1, \cdots, e_N\}$ denote the standard basis of $\mathbb{R}^N$. Consider the map $\pi : \mathbb{R}^N \to \mathbb{R}^n = \mathfrak{g}$ given by $\pi(\sum a_i e_i) = -\sum a_i \eta_i$. Since Δ is strongly convex, π is surjective. Moreover by smoothness, π maps $\mathbb{Z}^N$ onto $\mathbb{Z}^n$. Hence π induces the surjective map $\tilde{\pi} : T^N = \mathbb{R}^N/(2\pi\mathbb{Z})^N \to \mathbb{R}^n/(2\pi\mathbb{Z})^n = T^n$ and an injective dual map $\pi^* : (\mathbb{R}^n)^* \to (\mathbb{R}^N)^*$. Let K be the kernel of $\tilde{\pi}$, and $\mathfrak{k}$ the Lie algebra of K. Now $K = \mathfrak{k}/(2\pi\mathbb{Z})^N$ holds by smoothness of Δ and so K is connected. Then we have the following three exact sequences:

$$0 \longrightarrow \mathfrak{k} \xrightarrow{\ i\ } \mathbb{R}^N \xrightarrow{\ \pi\ } \mathbb{R}^n \longrightarrow 0, \tag{2.5}$$

$$0 \longrightarrow K \xrightarrow{\ i\ } T^N \xrightarrow{\ \tilde{\pi}\ } T^n \longrightarrow 0, \tag{2.6}$$

$$0 \longrightarrow (\mathbb{R}^n)^* \xrightarrow{\ \pi^*\ } (\mathbb{R}^N)^* \xrightarrow{\ i^*\ } \mathfrak{k}^* \longrightarrow 0 \tag{2.7}$$

where i is the inclusion map.

Now consider $\mathbb{C}^N$ with symplectic form $\sqrt{-1}\sum dz_i \wedge d\overline{z_i}$, and standard Hamiltonian action of T^N given for $t = (t_1, \cdots, t_N) \in (\mathbb{R}/2\pi\mathbb{Z})^N$ and $z = (z_1, \cdots, z_N)$ by $t \cdot z = (e^{t_1\sqrt{-1}} z_1, \cdots, e^{t_N\sqrt{-1}} z_N)$. Then we can take the moment map of this action as follows:

$$\sigma : \mathbb{C}^N \to (\mathbb{R}^N)^*, z \mapsto (\|z_1\|^2, \cdots, \|z_N\|^2) - (\kappa_1, \cdots, \kappa_N).$$

The action of K on $\mathbb{C}^N$ is induced by the restriction of the action of T^N. Moreover the moment map of this K-action is $i^* \circ \sigma$.

Let $Z = (i^* \circ \sigma)^{-1}(0)$ be the zero-level set. In fact, Z is compact and K acts on Z freely. Then we get a compact connected $2n$-dimensional symplectic manifold $M_\Delta := Z/K$ by the symplectic reduction. Let ω_Δ be the reduced symplectic form. Moreover M_Δ is toric and its moment image is Δ. For more details on this see [C].

Remark 2.5. *We can also apply Delzant construction to strongly convex good cones (cf.*[L3]*). But we can not apply this construction to weakly convex unimodular sets, especially weakly convex good cones. To construct corresponding manifolds in weakly convex cases, we use symplectic cuts, which we explain below.*

2.3. **Symplectic cuts.** Next, we review symplectic cuts due to Lerman [L1].

Definition 2.6 (symplectic cuts)**.** *Let (M,ω) be a symplectic manifold with symplectic form ω and suppose that the circle S^1 acts on M in the Hamiltonian way with moment map $\phi : M \to \mathbb{R}$. Now we take a symplectic manifold $(M \times \mathbb{C}, \omega + \sqrt{-1}dz \wedge d\overline{z})$ with the diagonal S^1-action. The moment map of this S^1-action is $\mu = \phi + \|z\|^2$. If S^1 acts freely on $\phi^{-1}(\kappa)$, then $\mu^{-1}(\kappa)/S^1$ is nonsingular and becomes a symplectic manifold. We call this the* ***symplectic cut of M with respect to the ray*** $(-\infty, \kappa]$ *and denote this by M^{κ}_{cut}.*

Here we considered S^1 as $\mathbb{R}/2\pi\mathbb{Z}$ so that the moment map of standard S^1-action on $\mathbb{C}$ is $\|z\|^2$.

Remark 2.7. *Since $\mu^{-1}(\kappa) = \{(m, z) \in M \times \mathbb{C} \mid \kappa - \phi(m) = \|z\|^2 > 0\} \sqcup \{(m, 0) \in M \times \mathbb{C} \mid \phi(m) = \kappa\}$ and both parts are S^1-invariant, $M^{\kappa}_{cut} = \mu^{-1}(\kappa)/S^1$ is the disjoint union of the quotient of these two parts. If we put $M_\kappa := \{(m, 0) \in M \times \mathbb{C} \mid \phi(m) = \kappa\}/S^1$ and $M^{\kappa}_0 := \{(m, z) \in M \times \mathbb{C} \mid \kappa - \phi(m) = \|z\|^2 > 0\}/S^1$ then we have $M^{\kappa}_{cut} = M_\kappa \sqcup M^{\kappa}_0$.*

Now consider the map

$$\sigma : \phi^{-1}((-\infty, \kappa)) \to M^{\kappa}_0, m \mapsto [m, \sqrt{\kappa - \phi(m)}] \tag{2.8}$$

where $[m, z]$ is the equivalence class of $(m, z) \in \mu^{-1}(\kappa)$.
One can easily see that σ is a symplectomorphism (see [L2]*, Theorem 2.5), so M^{κ}_0 can be identified with $\phi^{-1}(-\infty, \kappa)$, hence we can consider that M^{κ}_0 is an open symplectic submanifold of M in that sense. Nevertheless, in general, σ is not a Kähler isometry, that is, M^{κ}_0 is not embedded in M as a Kähler submanifold by σ.*

2.4. **Construction with symplectic cuts.** Given a unimodular set $\Pi = \Delta = \{x \in (\mathbb{R}^n)^* \mid \langle x, \eta_i\rangle \leqq \kappa_i, i = 1, \cdots, N\}$, we construct the symplectic toric manifold which corresponds to Δ. Here we are not assuming Δ is compact, and thus it is not necessarily a Delzant polytope.

Step 0: Let $M := T^*T^n = \mathbb{R}^n \times T^n$. We take a standard coordinate $(\mathbb{R}^n \times T^n, x_1, \cdots, x_n, \theta_1, \cdots, \theta_n)$ and let $\omega := \sum dx_i \wedge d\theta_i$. Moreover we take the canonical T^n-action on M, that is,

$$(t_1, \cdots, t_n) \cdot (x_1, \cdots, x_n, \theta_1, \cdots, \theta_n) = (x_1, \cdots, x_n, \theta_1 + t_1, \cdots, \theta_n + t_n)$$

and the moment map with this action is

$$\Phi : M \to (\mathbb{R}^n)^*, (x, \theta) \mapsto x,$$

where $x = (x_1, \cdots, x_n), \theta = (\theta_1, \cdots, \theta_n)$. Indeed, $\Phi(M) = (\mathbb{R}^n)^*$.
Step 1: We construct the symplectic toric manifold which corresponds to $\{x \in (\mathbb{R}^n)^* \mid \langle x, \eta_1\rangle \leqq \kappa_1\}$.
We consider $(M \times \mathbb{C}, \omega + \sqrt{-1}dz \wedge d\overline{z})$ and the diagonal S^1-action on this, but the S^1-action on M is the action of $\mathbb{R}\text{-span}\{\eta_1\}/2\pi\mathbb{Z} \subset T^n$, and the S^1-action on $\mathbb{C}$ is the standard action of $S^1 \subset \mathbb{C}$. Then the moment map μ_1 of this action is

$$\mu_1 : M \times \mathbb{C} \to \mathbb{R}^*, ((x, \theta), z) \mapsto \langle x, \eta_1\rangle + \|z\|^2 \tag{2.9}$$

Since the S^1-action is free on $\langle\Phi, \eta_1\rangle^{-1}(\kappa_1)$, we can take the symplectic cut $M_1 := \mu_1^{-1}(\kappa_1)/S^1$. Now we extend the T^n-action on M to $M \times \mathbb{C}$ as the trivial action on the second factor $\mathbb{C}$ and take the product $T^n \times S^1$-action on $M \times \mathbb{C}$. The moment map Ψ of this product action is

$$\Psi = \Phi \oplus \mu_1 : M \times \mathbb{C} \to (\mathbb{R}^n)^* \oplus \mathbb{R}^*, (x, \theta, z) \mapsto (x, \langle x, \eta_1\rangle + \|z\|^2). \tag{2.10}$$

Since Φ is S^1-invariant, we have the moment map Φ_1 with T^n-action on M_1 which is induced by T^n-action on $M \times \mathbb{C}$ and the following diagram commutes:

$$\begin{tikzcd}
\mu_1^{-1}(\kappa_1) \arrow[r, "\mathrm{inc}"] \arrow[d, "\mathrm{quot}"'] & M \times \mathbb{C} \arrow[r, "\Psi"] \arrow[d, "\Phi\oplus 0"] & (\mathbb{R}^n)^* \oplus \mathbb{R}^* \arrow[dl, "\mathrm{proj}"] \\
M_1 \arrow[r, "{}^{\exists}\Phi_1"'] & (\mathbb{R}^n)^* &
\end{tikzcd}$$

To compute $\Phi_1(M_1)$ we remark the following:

- $\Phi_1(M_1) = \Phi_1 \circ (quot)(\mu^{-1}(\kappa_1)) = (proj) \circ \Psi(\mu_1^{-1}(\kappa_1)) = (\Phi \oplus 0)(\mu_1^{-1}(\kappa_1))$,
- $\mu_1^{-1}(\kappa_1) = \{(x, \theta, z) \mid \langle x, \eta_1 \rangle + \|z\|^2 = \kappa_1\} = \{(x, \theta, z) \mid \langle x, \eta_1 \rangle \leqq \kappa_1, \|z\| = \sqrt{\kappa_1 - \langle x, \eta_1 \rangle}\}$.

Therefore $\Phi_1(M_1) = \{x \in (\mathbb{R}^n)^* \mid \langle x, \eta_1 \rangle \leqq \kappa_1\}$ and we get the symplectic toric manifold (M_1, ω_1, ϕ_1) where ω_1 is symplectic form which is induced by ω. One can easily see that a point in the inverse image $\Phi_1^{-1}(\{\langle x, \eta_1 \rangle = \kappa_1\})$ is fixed by the subgroup S_1 in T^n where $S_i := \mathbb{R}\text{-span}\{\eta_i\}/2\pi\mathbb{Z}$.

Step $2, \cdots, N$: We repeat the cutting as above for $S_2, \cdots S_N$, then we get the symplectic toric manifold (M_N, ω_N, Φ_N) with moment image Δ.

Let (M_k, ω_k, μ_k) be a symplectic toric manifold (orbifold) resulting from the k-times cuttings. We remark that, when we do this cut by the S_{k+1}-action, we have the following equivalence:

- $(M_k)_{\kappa_{k+1}}$ is non-singular $\iff$ The S_{k+1}-action is free on $\langle \Phi_k, \eta_{k+1} \rangle^{-1}(\kappa_{k+1})$.

Hence to be convinced that we can get a manifold at the end of this construction, we must prove S_{k+1}-action is free on $\Phi_k^{-1}(\Delta \cap F_{k+1})$ where F_{k+1} is the $(k+1)$-th facet of Π. Let p be a point in $\Phi_k^{-1}(\Delta \cap F_{k+1})$ and we take a maximal subset $I \in \{J \subset \{1, \cdots, k\} \mid \Phi_k(p) \in F_J\}$ where F_J is as given in Definition 2.1 and where maximality is given by the inclusion property. Then $p \in \Phi_k^{-1}(\Phi_k(p)) = T^n \cdot p = \{\Phi_k(p)\} \times (T^n/\Pi_{i \in I} S_i) \subset M_k$ since $\Phi_k(p) \in F_I$. The S_{k+1}-action on this set is given as below:

$$[t\eta_{k+1}] \cdot [x_1, \cdots, x_n, \theta_1, \cdots, \theta_n] = [x_1, \cdots, x_n, \theta_1 + t\eta_{k+1}^1, \cdots, \theta_n + t\eta_{k+1}^n] \tag{2.11}$$

where $t \in \mathbb{R}$ and $[\cdots]$ represent suitable equivalence classes.

If $[t_{k+1}\eta_{k+1}] \cdot [x, \theta] = [x, \theta]$ then

$$[t_{k+1}\eta_{k+1}] \in \Pi_{i \in I} S_i \iff {}^\exists[\sum_{i \in I} t_i \eta_i] + [t_{k+1}\eta_{k+1}] = 0 \iff \sum_{i \in I \cup \{k+1\}} t_i \eta_i \in 2\pi\mathbb{Z}^n. \tag{2.12}$$

Since Δ is unimodular, $\{\eta_i\}_{i \in I \cup \{k+1\}}$ is $\mathbb{Z}$-basis of the integral lattice of subtorus, so each t_i is a point in $2\pi\mathbb{Z}$. In particular $t_{k+1} \in 2\pi\mathbb{Z}$, then $[t_{k+1}\eta_{k+1}] = 0$ in S_{k+1}. Therefore the S_{k+1}-action is free on $\phi_k^{-1}(\Delta \cap F_{k+1})$. The S_{k+1}-action may not be free on $\Phi_k^{-1}(\mathbb{R}^n \setminus \Delta)$ (but that is locally free, that is, these points have the non-trivial discrete stabilizer), and note that a point with the non-trivial discrete stabilizer yields a orbifold point by the group reduction. Hence a manifold M_{k+1} that appears in the middle of the construction is an orbifold in generic cases. Nevertheless M_N has no singular point finaly since Δ is unimodular and so all orbifold points are cut off by the end of the last cutting, therefore that is a manifold.

We can reformulate N-times reduction as above to reduction at a time by using the reduction of product groups below:

Theorem 2.8 (cutting construction)**.** *Let $\Pi = \Delta := \{x \in (\mathbb{R}^n)^* \mid \langle x, \eta_i \rangle \leqq \kappa_i, i = 1, \cdots, N\}$ be a unimodular set and $(T^*T^n \times \mathbb{C}^N, \sum_i dx_i \wedge d\theta_i + \sqrt{-1}\sum_{i=1}^N dz_i \wedge d\overline{z}_i, T^n \times T^N, \Phi \oplus \mu)$ be a Hamiltonian $T^n \times T^N$-space as follows:*
$T^n \times T^N$-action on $T^*T^n \times \mathbb{C}^N$ is given by

$$(s_1, \cdots, s_n, t_1, \cdots, t_N) \cdot (x, \theta, z_1, \cdots, z_N) = (x, \theta + \sum_{i=1}^n s_i e_i + \sum_{i=1}^N t_i \eta_i, e^{\sqrt{-1}t_1} z_1, \cdots, e^{\sqrt{-1}t_N} z_N) \tag{2.13}$$

where (x, θ) is action-angle coordinates on $T^*T^n \cong \mathbb{R}^n \times \mathbb{R}^n/2\pi\mathbb{Z}^n$, and e_i is the standard basis of $\mathbb{R}^n$. Moreover the moment map of this action is

(2.14)

$$\Phi \oplus \mu : T^*T^n \times \mathbb{C}^N \to (\mathbb{R}^n \oplus \mathbb{R}^N)^*, (x, \theta, z_1, \cdots, z_N) \mapsto x \oplus (\langle x, \eta_1 \rangle + \|z_1\|^2 - \kappa_1, \cdots, \langle x, \eta_N \rangle + \|z_N\|^2 - \kappa_N).$$

Now we consider the symplectic quotient $M_\Delta := \mu^{-1}(0)/T^N$, then there are the induced symplectic form ω_Δ, the induced T^n-action and the induced moment map Φ_Δ on M_Δ. Moreover $\mathrm{Im}\Phi_\Delta = \Delta$ holds.

Remark 2.9. *If Δ is not closed i.e. $\Delta = U \cap \{x \in (\mathbb{R}^n)^* \mid \langle x, \eta_i \rangle \leqq \kappa_i, i = 1, \cdots, N\}$ where U is open set of $(\mathbb{R}^n)^*$, then take $U \times T^n \times \mathbb{C}^N$ instead of $T^*T^n \times \mathbb{C}^N$ and construct M_Δ as above.*

Remark 2.10. *This construction is an explicit version of collapsing construction (*[KL]*) for unimodular sets. If Δ is a Delzant polytope, Theorem 2.8 also means that the Delzant construction is a symplectic cut of T^*T as a symplectic geometric operation, which has been pointed out in* [MT].

Remark 2.11. *By the property of symplectic cutting (cf.Remark 2.7), we have the action-angle coordinates on* $M_0 := \Phi_\Delta^{-1}(\mathring{\Delta})$ *in the straightforward manner. Let* $(T^*T^n, \sum_i dx_i \wedge d\theta_i, T^n, \Phi)$ *and* $(M_\Delta, \omega_\Delta, T^n, \Phi_\Delta)$ *be as above. Note that* $M_0 = \{(x, \theta, z_1, \cdots, z_N) \in T^*T^n \times \mathbb{C}^N \mid \|z_i\|^2 = \kappa_i - \langle x, \eta_i\rangle > 0, i = 1, \cdots, N\}/T^N$ *and now consider the map*

$$\sigma : \Phi^{-1}(\mathring{\Delta}) \to M_0,\ (x, \theta) \mapsto [x, \theta, \sqrt{\kappa_1 - \langle x, \eta_1\rangle}, \cdots, \sqrt{\kappa_N - \langle x, \eta_N\rangle}] \tag{2.15}$$

where $[\cdot,\cdot]$ *is the equivalence class of the* T^N*-quotient.*
One can easily see that σ *is an equivariant symplectomorphism. Moreover the canonical coordinates* (x, θ) *of* T^*T^n *is the action-angle coordinates. Therefore we can obtain the action-angle coordinates on* M_0 *by pulling back the canonical coordinates* (x, θ) *of* T^*T^n *by* σ^{-1}*. Hence in what follows we take them as the canonical coordinates. We can identified* M_0 *with* $\Phi^{-1}(\mathring{\Delta})$ *as a symplectic manifold and we can also consider that* M_0 *is an open symplectic submanifold of* T^*T^n *in that sense.*

3. The diffeomorphism type and the fundamental group of M_Δ

This section deals with diffeomorphism types and homotopy groups of M_Δ which corresponds to weakly convex unimodular sets.

Theorem 3.1. *Let* X *be a* $2n$*-dimensional toric variety with fan* $(\Sigma, \mathbb{Z}^n)$*. Then there exists a* $2(n-k)$*-dimensional toric variety* B *such that* $X = (\mathbb{C}^*)^k \times B$ *if and only if there exists a* $(n-k)$*-dimensional sub lattice* N *of* $\mathbb{Z}^n$ *such that* $\Sigma \subset \mathbb{R}\text{-span}(\mathrm{N})$.

Proof. See the book of Fulton([F]) p22 exercise. □

We can translate the above theorem in the complex situation to the symplectic situation as a below:

Theorem 3.2. *Let* $\Delta = \{x \in (\mathbb{R}^n)^* \mid \langle x, \eta_i\rangle \leqq \kappa_i, i = 1, \cdots, N\}$ *be a weakly convex unimodular set and suppose that* $\mathrm{codim}(\mathbb{R}\text{-span}\{\eta_1, \cdots, \eta_\mathrm{N}\}) = \mathrm{k}$*. Then we can choose* $\{i_1, \cdots, i_{n-k}\} \subset \{1, \cdots, n\}$ *such that* $\eta_i' := {}^t(\eta_i^{i_1}, \cdots, \eta_i^{i_{n-k}}) \in \mathbb{R}^{n-k}$ *for each* $i = 1, \cdots, N$ *and* $\Delta' := \{x \in (\mathbb{R}^{n-k})^* \mid \langle x, \eta_i'\rangle \leqq \kappa_i, i = 1, \cdots, N\}$ *is strongly convex. Moreover* M_Δ *diffeomorphic to* $(\mathbb{C}^*)^k \times M_{\Delta'}$*.*

Proof. Set $B = M_{\Delta'}$ and take sublatice $\mathbb{Z}\text{-span}\{\eta_1, \cdots, \eta_\mathrm{N}\}$ as N in above theorem then it is clear by the sufficient part of that. □

Corollary 3.3. *Let* $\Delta = \{x \in (\mathbb{R}^n)^* \mid \langle x, \eta_i\rangle \leqq \kappa_i, i = 1, \cdots, N\}$ *be a weakly convex unimodular set with* $\mathrm{codim}(\mathbb{R}\text{-span}\{\eta_1, \cdots, \eta_\mathrm{N}\}) = \mathrm{k}$*. Then the following holds:*

$$\pi_m(M_\Delta) = \begin{cases} \pi_m(M_{\Delta'}) & (m \geqq 2) \\ \mathbb{Z}^k \times \pi_m(M_{\Delta'}) & (m = 1) \\ \{0\} & (m = 0) \end{cases}$$

Hence if we want to know the homotopy groups of M_Δ *then we have only to consider the homotopy groups of its strongly convex part* $M_{\Delta'}$*. For example, the* 1*-dimensional and* 2*-dimensional homotopy groups of the strongly convex good cone were computed in E.Lerman* [L4].

4. Kähler Cuts

Now, T^*T^n have the natural Kähler structure, i.e. it have the standard compatible almost complex structure J_{ST} as the following. We take coordinates $(T^*T^n, x_1, \cdots, x_n, \theta_1, \cdots, \theta_n)$, then

$$J_{ST} : T^*M \to T^*M,\ J_{ST}(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial \theta_i}, J_{ST}(\frac{\partial}{\partial \theta_i}) = -\frac{\partial}{\partial x_i} \tag{4.1}$$

where M is T^*T^n.

Hence M_Δ has the canonical Kähler structure that is induced by the standard Kähler structure of T^*T^n and the above construction. Now we compute the induced Kähler structure of M_Δ by applying the following proposition.

Proposition 4.1. *Let* $\Delta = \{x \in (\mathbb{R}^n)^* \mid \langle x, \eta_i\rangle \leqq \kappa, i = 1, \cdots, N\}$ *be a unimodular set and let* $(M_\Delta, \omega_\Delta, T^n, \Phi_\Delta)$ *be the Hamiltonian space as in Theorem 2.8. Then there exists a biholomorphic map* g *between* $(M_\Delta \supset) M_0 = \Phi_\Delta^{-1}(\mathring{\Delta})$ *with the induced complex structure and* T^*T^n *with the standard complex structure which can be expressed in terms of the action angle coordinates, and such a biholomorphism* g *is given by the following formula*

$$g : M_0 \to T^*T^n,\ (x,\theta) \mapsto (x - \frac{1}{2}\sum_{i=1}^{N}[\log(\kappa_i - \langle x, \eta_i\rangle)]\eta_i, \theta) \tag{4.2}$$

where $x = (x_1, \cdots, x_n)$, $\theta = (\theta_1, \cdots, \theta_n)$ and since the symplectic form induces the natural isomorphism $\mathfrak{t}^n \cong \mathbb{R}^n \cong \mathfrak{t}^{n*}$ we may consider η_i and x in the same space.

Proof. For simplicity, we set $(M,\omega) = (T^*T^n, \sum_i dx_i \wedge d\theta_i)$. We first note the following.
(1) Let (x,θ) be a point of M_0 and suppose $t = (t_1, \cdots, t_N) \in \mathbb{R}^N$ satisfies

$$(\kappa_1, \cdots, \kappa_N) - (e^{2t_1}, \cdots, e^{2t_N}) = \phi(x,\theta) = (\langle x, \eta_1\rangle, \cdots, \langle x, \eta_N\rangle),$$

then $\exp(\sum_{i=1}^N t_i \nabla\phi_i)((y,\xi)) = (x,\theta)$ if and only if $(y,\xi) = g(x,\theta)$ where ϕ_i is the i-th component of ϕ, that is $\langle x, \eta_i\rangle$.

We prove g is a biholomorphism from M_0 to its image. To see this, we consider the following biholomorphic map

$$f : M \times (\mathbb{C}^*)^N \to M \times (\mathbb{C}^*)^N,\ (p, z) \mapsto (z \cdot p, z) \tag{4.3}$$

where $z = (z_1, \cdots, z_N) = (e^{r_1+\sqrt{-1}s_1}, \cdots, e^{r_N+\sqrt{-1}s_N})$, $p = (x,\theta)$, and $z \cdot p := (x + \Sigma_i r_i\eta_i, \theta + \Sigma_i s_i\eta_i)$.

Give a diagonal action of $(\mathbb{C}^*)^N$ on $M \times (\mathbb{C}^*)^N$. Now the pullback by f of the Kähler form $\omega + \sqrt{-1}\sum_{i=1}^N dz_i \wedge d\overline{z_i}$ is invariant under the diagonal action of that $T^N(\subset (\mathbb{C}^*)^N)$. Moreover the diagonal T^N-action is the Hamiltonian action with a moment map $\overline{\mu}(p,z) = \phi(z \cdot p) + (\|z_1\|^2, \cdots, \|z_N\|^2) - (\kappa_1, \cdots, \kappa_N)$. In particular, f maps the level set $\overline{\mu}^{-1}(0)$ to $\mu^{-1}(0)$, where μ is the same map as in Theorem 2.8 and induce the Kähler isometry

$$h : \overline{\mu}^{-1}(0)/T^N \to \mu^{-1}(0)/T^N. \tag{4.4}$$

To describe this isometry more explicitly, take the following subsets

$$\{[(x,\theta), (e^{t_1}, \cdots, e^{t_N})] \mid \langle x + \sum_{k=1}^N t_k\eta_k, \eta_i\rangle + e^{2t_i} = \kappa_i, i = 1, \cdots, N\}(\subset \overline{\mu}^{-1}(0)/T^N), \tag{4.5}$$

$$M_0 = \{[(x,\theta), (e^{t_1}, \cdots, e^{t_N})] \mid \langle x, \eta_i\rangle + e^{2t_i} = \kappa_i, i = 1, \cdots, N\}(\subset \mu^{-1}(0)/T^N) \tag{4.6}$$

where $[\cdot,\cdot]$ represents suitable equivalence classes.

Suppose h maps a point $[(y,\xi), (e^{s_1}, \cdots, e^{s_N})]$ in (4.5) to a point $[(x,\theta), (e^{t_1}, \cdots, e^{t_N})]$ in (4.6), then

$$s_i = t_i,\ \theta = \xi,\ x = y + \sum_{k=1}^N t_k\eta_k,\ \textit{and}\ e^{2t_i} = \kappa_i - \langle x, \eta\rangle\ (\text{for } i = 1, \cdots, N).$$

Therefore $(y,\xi) = g(x,\theta)$ holds since (1) and $\exp(\sum_{i=1}^N t_i\nabla\phi_i)((y,\xi)) = (x,\theta)$ iff $x = y + \sum_{k=1}^N t_k\eta_k$ and $\theta = \xi$. That indicates $h = g^{-1}$. Therefore $g : M_0 \to \mathrm{Im}(g)$ is a biholomorphim between (4.5) and (4.6) with the induced complex structures. Note that open set (4.5) have the same complex structure as M by GIT-quotient.

Next we show g is surjective. To see this, we take the following map:

$$\tilde{g} : \mathring{\Delta} \to (\mathbb{R}^n)^*, x \mapsto x - \frac{1}{2}\sum_{i=1}^N \log(\kappa_i - \langle x, \eta_i\rangle)\eta_i \tag{4.7}$$

where $\mathring{\Delta}$ denotes the interior of Δ.

Then, the following diagram is commutative:

$$\begin{array}{ccc} M_0 & \xrightarrow{g} & M \\ \downarrow{\scriptstyle\Phi} & & \downarrow{\scriptstyle\Phi} \\ \mathring{\Delta} & \xrightarrow{\tilde{g}} & (\mathbb{R}^n)^* \end{array}$$

where, we must consider $M_0 = \Phi^{-1}(\mathring{\Delta}) \subset M$.
It is clear that g is surjective iff $\tilde{g}$ is surjective. Hence the remains of this proof is to prove $\mathrm{Im}(\tilde{g}) = (\mathbb{R}^n)^*$.

To see this, we remark $\tilde{g}$ is diffeomorphism to its image because g is diffeomorphism to its image. As a result, $\mathrm{Im}(\tilde{g})$ is an n-dimensional manifold in $(\mathbb{R}^n)^*$. In particular, $\mathrm{Im}(\tilde{g})$ is open. Suppose $\mathrm{Im}(\tilde{g})$ have some boundary points, then that is corresponding to the boundary of Δ since $\mathrm{Im}(\tilde{g})$ is homeomorphic to Δ. However definition of $\mathrm{Im}(\tilde{g})$ shows that the boundary of Δ is mapped on to infinity. Therefore the boundary of $\mathrm{Im}(\tilde{g})$ is empty, that is, $\tilde{g}$ is surjective. □

Remark 4.2. *This proposition is the explicit version of Kähler cuts in* [BGL].

Proof of Theorem 1.1 We compute the Kähler structure $(\omega_\Delta, J_\Delta, g_\Delta)$ of M_Δ on M_0 by applying Proposition 4.1. First of all, note that we have already get action-angle coordinates (x,θ) on M_0 by the symplectic cutting construction in Remark 2.11 sense. Hence we describe the complex structure J_Δ and the metric g_Δ in action-angle coordinates, i.e, we consider a tangent space of $M_0 = \mathbb{R}$-span$\{\frac{\partial}{\partial \mathrm{x}_1}, \cdots, \frac{\partial}{\partial \mathrm{x}_\mathrm{n}}, \frac{\partial}{\partial \theta_1}, \cdots, \frac{\partial}{\partial \theta_\mathrm{n}}\} \cong$ a tangent space of T^*T^n. Now we note that ω, J_{ST} and the derivation of the biholomorphism g is represented by the following matrices:
we set

$$\eta_1 = \begin{pmatrix} \eta_1^1 \\ \cdot \\ \cdot \\ \cdot \\ \eta_1^n \end{pmatrix}, \cdots, \eta_N = \begin{pmatrix} \eta_N^1 \\ \cdot \\ \cdot \\ \cdot \\ \eta_N^n \end{pmatrix}, l_1(x) := \kappa_1 - \langle x, \eta_1 \rangle, \cdots, l_N(x) := \kappa_N - \langle x, \eta_N \rangle, l_\infty(x) := \sum l_i(x),$$

and

$$G = \begin{pmatrix} 1 + \frac{1}{2}\sum \frac{(\eta_i^1)^2}{l_i(x)} & \frac{1}{2}\sum_i \frac{\eta_i^1 \eta_i^2}{l_i(x)} & \cdots & \frac{1}{2}\sum_i \frac{\eta_i^1 \eta_i^n}{l_i(x)} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \frac{1}{2}\sum \frac{\eta_i^1 \eta_i^n}{l_i(x)} & \cdots & \frac{1}{2}\sum_i \frac{\eta_i^{n-1} \eta_i^n}{l_i(x)} & 1 + \frac{1}{2}\sum_i \frac{(\eta_i^n)^2}{l_i(x)} \end{pmatrix}.$$

Then,

$$\omega = \begin{pmatrix} O & I \\ -I & O \end{pmatrix}, J_{ST} = \begin{pmatrix} O & -I \\ I & O \end{pmatrix}, g_* = \begin{pmatrix} G & O \\ O & I \end{pmatrix}. \tag{4.8}$$

Then, since $\omega_\Delta = \omega$, $J_\Delta = g_*^{-1} \circ J_{ST} \circ g_*$ and $g_\Delta(\cdot,\cdot) = \omega_\Delta(\cdot, J_\Delta \cdot)$, they are represented by the following matrices:

$$\omega_\Delta = \begin{pmatrix} O & I \\ -I & O \end{pmatrix}, J_\Delta = \begin{pmatrix} O & -G^{-1} \\ G & O \end{pmatrix}, g_\Delta = \omega_\Delta(\cdot, J_\Delta \cdot) = \begin{pmatrix} G & O \\ O & G^{-1} \end{pmatrix}. \tag{4.9}$$

Now we wish to take a function, denoted by Sp, whose Hessian matrix provides G. By a direct calculation one can show that this function $\mathrm{Sp} : \mathring{\Delta} \to \mathbb{R}$ is given by the following formula:

$$\mathrm{Sp}(x) := \frac{1}{2}\|x\|^2 + \frac{1}{2}\sum l_i(x) \log l_i(x) - \frac{1}{2} l_\infty(x). \tag{4.10}$$

We call this function the *symplectic potential* of the canonical Kähler structure. This completes the proof of Theorem 1.1.

Remark 4.3. *A symplectic potential is introduced by V.Guillemin* ([G]) *as the Legendre transformation of a Kähler potential. In our situation, the function* Sp *is determined by the construction rather than the Legendre transformation, but, nevertheless we can call it the symplectic potential for similarity of between representation (4.9) and usual representation of the Kähler structure on action-angle coordinates (c.f.*[A]*, p.*7*).*

Remark 4.4. *If we drop $l_\infty(x)$ from* $\mathrm{Sp}(x)$*, it define the same Kähler structure as* (4.9)*, but its derivation is not equal to $\tilde{g}$. Similarly there are other deformations of* Sp *whose Hessian is non-degenerate. Specifically if Δ is polytope, then $\mathrm{Sp}(x) - \frac{1}{2}\|x\|^2$ coincide with the well known Guillemin's symplectic potential. On the other hand, if Δ is weakly convex, then we can not drop $\frac{1}{2}\|x\|^2$ from* $\mathrm{Sp}(x)$ *because if we drop it, the Hessian is degenerate, that is, it does not define Kähler structure.*

Remark 4.5. *If a unimodular set $\mathbb{R}^n \supset \Delta$ is the n-simplex then the fixed point of $\tilde{g}$ coincides with the barycenter.*

In one-time cutting cases, we can compute G^{-1} explicitly:

$$G^{-1} = \frac{1}{1+\frac{\|\eta\|^2}{\kappa-\langle x,\eta\rangle}} \begin{pmatrix} 1+\frac{1}{2}\frac{\sum_{i\neq 1}(\eta^i)^2}{\kappa-\langle x,\eta\rangle} & -\frac{1}{2}\frac{\eta^1\eta^2}{\kappa-\langle x,\eta\rangle} & \cdots & -\frac{1}{2}\frac{\eta^1\eta^n}{\kappa-\langle x,\eta\rangle} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ -\frac{1}{2}\frac{\eta^1\eta^n}{\kappa-\langle x,\eta\rangle} & \cdots & -\frac{1}{2}\frac{\eta^{n-1}\eta^n}{\kappa-\langle x,\eta\rangle} & 1+\frac{1}{2}\frac{\sum_{i\neq n}(\eta^i)^2}{\kappa-\langle x,\eta\rangle} \end{pmatrix}. \tag{4.11}$$

5. APPLICATION TO CONTACT TORIC MANIFOLDS

Here we discuss the case when a unimodular set Δ is a polyhedral cone. Then M_Δ is a symplectic cone and it include the contact toric manifold $\underline{M_\Delta}$ as a sub-manifold which is a pre-image of the unit sphere with the moment map image. Here, by a symplectic cone, we mean is a symplectic manifold (S,ω) with a free proper action $\{\rho_t\}_{t\in\mathbb{R}}$ of the real line such that $\rho_t^*\omega = e^t\omega$ for all $t \in \mathbb{R}$. Now we have already gotten the Kähler structure on M_Δ, then it is reasonable to ask "is that sub-manifold Sasakian or not?"

Remark 5.1. *It is clear that the above canonical Kähler metric is not cone metric (c.f.* [MSY]*, p.44), hence it does not give $\underline{M_\Delta}$ a Sasakian structure, but it is left that a possibility of M_Δ have an another cone Kähler structure with respect to which $\underline{M_\Delta}$ is Sasakian.*

We prove the following theorem to solve this problem:

Theorem 5.2. *$\underline{M_\Delta}$ is of K-contact type if and only if the corresponding polyhedral unimodular cone Δ is strongly convex (see below for the precise definition of K-contact type).*

Generally speaking, if a contact manifold is of Sasakian type then it is also of K-contact type. Hence the above theorem says $\underline{M_\Delta}$ which corresponds to weakly convex cone is of non-Sasakian type.

5.1. Preliminaries.

We recall some basic notions of contact geometry. This and next subsections are based on [BG, Bl, Bo].

Let M be an orientable $(2n-1)$-dimensional manifold and suppose D is a co-dimension one subbundle of TM. Then the pair (M,D) is a contact manifold if and only if D is maximally non-integrable distribution, that is, $\mathcal{L}_X(\Gamma(D)) \nsubseteq \Gamma(D)$ holds for any non-zero $X \in \Gamma(D)$ where $\Gamma(D)$ is the space of sections of D. We call D the contact distribution. Let D^0 be the annihilator line bundle of D, and take a nowhere vanishing section α of $D^\circ \subset T^*M$. Then this 1-form α have the following properties:

1. $\alpha \wedge (d\alpha)^{n-1} \neq 0$, $D = \ker\alpha$;
2. there exists a unique vector field ξ such that $\alpha(\xi) = 1$ and $\iota_\xi d\alpha = 0$;
3. ξ generates a trivial line bundle L_ξ and the characteristic foliation $\mathcal{F}_\xi$;
4. L_ξ provides a splitting of tangent bundle $TM = D \oplus L_\xi$.

We call α satisfying (1) a **contact form** representing D and ξ satisfying (2) the **Reeb vector field** of α. Indeed, for any nowhere vanishing function f, $\alpha' := f\cdot\alpha$ is also a contact form of the same distribution D. On the other hand, if α and α' are contact forms of the same distribution then there exists nowhere vanishing function f such that $\alpha' := f\cdot\alpha$. Additionally, we determine the co-orientation of M by choosing the positive part of $D^\circ \setminus \{0\text{-}section\}$ and we denote this by D°_+. It is clear that $\alpha,\alpha' \in \Gamma(D^\circ_+) \iff {}^\exists f > 0$ s.t. $\alpha' = f\cdot\alpha$.

Definition 5.3. *Let $\alpha \in \Gamma(D^\circ_+)$. We say that an almost complex structure J is compatible with D if it satisfies the following condition for all $X, Y \in \Gamma(D)$*

$$d\alpha(JX,JY) = d\alpha(X,Y),\ d\alpha(JX,Y) > 0\,(X,Y \neq 0)$$

We denote by $\mathcal{J}(D)$ the set of all compatible almost complex structures with D.

Definition 5.4. *Let (M,D) be a contact manifold. A (co-dimension one) almost CR structure is a splitting of the complexified bundle $D\otimes\mathbb{C} = D^{1,0}\oplus D^{0,1}$ which satisfies $D^{1,0}\cap D^{0,1} = \{0\}$ and $D^{0,1} = \overline{D^{1,0}}$, where the bar denotes complex conjugation. When we have an almost CR structure we also have an endomorphism J of D such that $J^2 = -\mathrm{id}$. An endomorphism J (or equivalently almost CR structure) is said to be integrable if it satisfies two conditions:*

1. $[JX,Y]+[X,JY] \in \Gamma(D)$ $({}^\forall X,Y \in \Gamma(D))$;
2. $N(X,Y) = [JX,JY]-[X,Y]-J[JX,Y]-J[X,JY] = 0$.

Moreover, an almost CR structure J is said to be partially integrable if condition (1) *holds.*

Then we have the following proposition:

Proposition 5.5. *Let (M, D) be a contact manifold, and choose a compatible almost complex structure $J \in \mathcal{J}(D)$. Then the pair (D, J) defines a partially integrable strictly pseudo-convex almost CR structure.*

Proof. See [Bo]. □

For almost CR structure (M, D, J) we can extend the almost complex structure J to an endomorphism Φ by choosing a contact form $\alpha \in \Gamma(D^\circ_+)$ and setting $\Phi_{|D} := J$, $\Phi\xi := 0$. We call the pair (α, Φ) a polarized almost CR structure since this extension depends on the choice of $\alpha \in \Gamma(D^\circ_+)$.

Definition 5.6. *In general, a contact manifold (M, D) is said to have an almost contact structure if it admits a tensor field Φ of type (1,1), a vector field ξ, and a 1-form α satisfying*

$$\Phi^2 = -I + \alpha \otimes \xi, \ \alpha(\xi) = 1.$$

If M has a Remannian metric g satisfying $g(\Phi X, \Phi Y) = g(X, Y) - \alpha(X)\alpha(Y)$, we call a quadruplet (Φ, ξ, α, g) an **almost contact metric structure** *on M and g is said to be compatible.*

On the other hand a Riemannian metric g is an associated metric if, first of all,

$$\alpha(X) = g(X, \xi)$$

and secondly, there exists a tensor field Φ of type $(1, 1)$ *such that*

$$\Phi^2 = -I + \alpha \otimes \xi, \ d\alpha(X, Y) = g(X, \Phi Y).$$

Then we call a quadruplet (Φ, ξ, α, g) a **contact metric structure**.

In our situation, namely under the the conditions of Proposition 5.5, we can get a contact metric structure by defining,

$$g := g_D \oplus \alpha \otimes \alpha = d\alpha \circ (\Phi \otimes \mathrm{id}) + \alpha \otimes \alpha$$

for $\alpha \in \Gamma(D^\circ_+)$, $J \in \mathcal{J}(D)$, and a polarized almost CR structure Φ. We call this the canonical contact structure of (α, J). Fixing α there is a 1-1 correspondence between compatible almost complex structures on D and canonical contact metrics.

Definition 5.7. *A contact metric manifold $(M, \alpha, \xi, \Phi, g)$ is said to be a K-contact if ξ is a Killing vector field, that is, its flow constitutes a subgroup of isometries. Moreover, a K-contact manifold $(M, \alpha, \xi, \Phi, g)$ is Sasakian if (D, J) is integrable. Moreover, a contact manifold (M, D) is of* **K-contact type** *if it admits some K-contact structure, and a contact manifold (M, D) is of* **Sasakian type** *if it admits some Sasakian structure as well.*

Remark 5.8. *The definition of Sasakian manifold can be equivalently rephrased as follows. A K-contact manifold $(M, \alpha, \xi, \Phi, g)$ is Sasakian if the symplectic cone $(D^\circ_+, d(e^t\alpha), J_c)$ with the almost complex structure J_c satisfying*

$$J_c(X \oplus f\frac{\partial}{\partial t}) := (\Phi(X) - f\xi) \oplus \alpha(X)\frac{\partial}{\partial t}, \ (X \in \Gamma(TM), f \in C^\infty(D^\circ_+))$$

is a Kähler cone, that is, J_c is integrable (c.f. [BG]*).*

5.2. **The group of contactomorphisms.** Let $\mathfrak{Diff}(M)$ denote the group of diffeomorphisms of M. We regard $\mathfrak{Diff}(M)$ as a Fréchet Lie group with the compact-open C^∞ topology. For more details on this see [M, Ba, O, KM]. Since we deal almost exclusively with compact manifolds the compact-open C^∞ topology will suffice for our purposes.

Definition 5.9. *Let (M, D) be a closed connected contact manifold. Then we define the group $\mathfrak{Con}(M, D)$ of all* **contactomorphisms** *by*

$$\mathfrak{Con}(M, D) = \{\phi \in \mathfrak{Diff} \mid \phi_* D \subset D\}.$$

Further, we denote by $\mathfrak{Con}(M, D)^+$ the subgroup of $\mathfrak{Con}(M, D)$ that preserves the orientation of D. For any contact form α, it is easy to see that

$$\mathfrak{Con}(M, D) = \{\phi \in \mathfrak{Diff}(M) \mid \phi^*\alpha = f^\alpha_\phi \alpha, f^\alpha_\phi \in C^\infty(M)^*\}$$

and

$$\mathfrak{Con}(M, D)^+ = \{\phi \in \mathfrak{Diff}(M) \mid \phi^*\alpha = f^\alpha_\phi \alpha, f^\alpha_\phi \in C^\infty(M)^+\}$$

where $C^\infty(M)^*$ denotes the subset of nowhere vanishing functions in $C^\infty(M)$, and $C^\infty(M)^+$ denotes the subset of positive functions. Moreover we define the subgroup $\mathfrak{Con}(M,\alpha)$ of **strict contactomorphisms***, whose elements preserve a contact form α:*

$$\mathfrak{Con}(M,D)^+ = \{\phi \in \mathfrak{Diff}(M) \mid \phi^*\alpha = \alpha\}.$$

We can consider the Lie algebra of those groups as infinitesimal transformations,

$$\mathfrak{con}(M,D) := \{X \in \Gamma(TM) \mid \mathcal{L}_X\alpha = a(X)\alpha, a(X) \in C^\infty(M)\},\ \ \mathfrak{con}(M,\alpha) := \{X \in \Gamma(TM) | \mathcal{L}_X\alpha = 0\}.$$

Then,

Lemma 5.10.

$$\Gamma(D) \cap \mathfrak{con}(M,D) = \{0\}.$$

Proof. See [Bo]. □

If we choose a contact form $\alpha \in \Gamma(D_+^\circ)$ then we have a well known Lie algebra isomorphism between $\mathfrak{con}(M,D)$ and $C^\infty(M)$ with the Poisson-Jacobi bracket defined explicitly by $X \mapsto \alpha(X)$. Then one easily sees that the subalgebra $\mathfrak{con}(M,\alpha)$ is isomorphic to the subalgebra of F-invariant functions $C^\infty(M)^F$ where F is a flow of the Reeb vector field ξ of α. In particular, $\xi \in \mathfrak{con}(M,\alpha)$.

In fact, the centralizer of a Reeb vector field ξ in $\mathfrak{con}(M,D)$ is $\mathfrak{con}(M,\alpha)$. To see that we prove the following lemma:

Lemma 5.11. *Let (M,D) be a contact manifold and take a contact form $\alpha \in \Gamma(D_+^\circ)$, a Reeb vector field ξ of α, and $X \in \mathfrak{con}(M,D)$. Then the following holds;*

$$[\xi, X] = 0 \iff \xi(\alpha(X)) = 0.$$

Proof. Since $\alpha([\xi,X]) = \mathcal{L}_\xi\iota_X\alpha - \iota_X\mathcal{L}_\xi\alpha = \xi(\alpha(X))$, only if part is clear. If $\xi(\alpha(X)) = 0$ then $\alpha([\xi,X]) = 0$ and hence $[\xi,X] \in \Gamma(D)$ holds. On the other hand, since $[\xi,X] \in \mathfrak{con}(M,D)$ and Lemma 5.10, $[\xi,X] = 0$. □

Therefore the centralizer of the Reeb vector field ξ is a pre-image of $\ker\xi = C^\infty(M)^F$ by isomorphism, that is, $\mathfrak{con}(M,\alpha)$. Thus we have proven the following.

Lemma 5.12. *The centralizer of a Reeb vector field ξ in $\mathfrak{con}(M,D)$ is $\mathfrak{con}(M,\alpha)$.*

5.3. **Toric action.**

Definition 5.13. *Let (M,D) be a contact manifold, let T be a torus, and suppose T acts on M. Then a triplet (M,D,T) is a contact toric manifold if and only if the torus T is embedded in $\mathfrak{Con}(M,D)$ and $\dim T$ is equal to $\frac{1}{2}(\dim M + 1)$. Now fixing a contact form α the α-moment map Φ_α is defined by $\Phi_\alpha : M \to \mathfrak{t}^*$, $\langle\Phi_\alpha(x), X\rangle = \alpha(X)(x)$.*

We note that the α-moment map depends on a particular choice of a contact form and not just on the contact structure. Hence we take more *canonical* or *universal* moment map as follows:

- Note that we can take a T-invariant contact form $\alpha \in \Gamma(D_+^\circ)$ by averaging α over the torus T and hence we can assume $T \subset \mathfrak{Con}(M,\alpha)$. Since the image of arbitrary α'-moment map do not in fact contain the origin, we can take a normalized contact form $\alpha := \frac{\alpha'}{\|\Phi_{\alpha'}\|}$ and then $\|\Phi_\alpha\| \equiv 1$. We call this the canonical moment map.
- In order to treat the contact 1-forms on an equal footing, we consider D_+° as a symplectization of M and take a moment map $\Psi : D_+^\circ \to \mathfrak{t}^*$ given by $\langle\Psi(x,\alpha_x), X\rangle = \alpha(X)(x)$. We call this the universal moment map.

Definition 5.14. *Let (M,D,T,Ψ) be a contact toric manifold as above. We define the moment cone $C(\Psi)$ to be the set*

$$C(\Psi) := \mathrm{Im}\Psi \cup \{0\}.$$

Note that symplectic cone is non-compact toric symplectic manifold. Hence $\mathrm{Im}\Psi$ is a unimodular set. In particular we can denote $C(\Psi) = \{x \in \mathfrak{t}^* \mid \langle x, \eta_i\rangle \geqq 0, i = 1, \cdots, N\}$ for some $\eta_i \in \mathfrak{t}$, $i = 1, \cdots, N$.

Definition 5.15. *A polyhedral cone C is* **good** *if and only if $C \setminus \{0\}$ is a unimodular set.*

There is well known Lerman's classification theorem of c.c.c.t. (compact connected contact toric) manifolds [L3]. From this theorem, there is a one-to-one correspondence between c.c.c.t. manifolds with a non-free toric action and good cones.

A c.c.c.t. manifold $(M, \alpha, T, \Phi_\alpha)$ with the canonical contact form is embedded in the symplectic cone $(D_+^\circ, d(e^t\alpha))$ as a pre-image of the intersection of a moment cone $C(\Psi)$ and the unit sphere. In fact the symplectization commutes with the symplectic/contact cutting (see [L2]). Hence we can construct c.c.c.t. manifolds by the cutting construction similarly to the symplectic case as follows:

Theorem 5.16. *Let $\Delta := \{x \in (\mathbb{R}^n)^* \setminus \{0\} \mid \langle x, \eta_i \rangle \geqq 0, i = 1, \cdots, N\}$ be a unimodular cone and $(S^*(T^n) \times \mathbb{C}^N, \sum_{i=1}^n x_i d\theta_i + \frac{\sqrt{-1}}{2} \sum_{i=1}^N (z_i d\overline{z}_i - \overline{z_i} dz_i), T^n \times T^N, \Phi \oplus \mu)$ be a Hamiltonian $T^n \times T^N$-space described as follows:*
$T^n \times T^N$-action on $S^*(T^n) \times \mathbb{C}^N$ is given by

$$(5.1)\quad (s_1, \cdots, s_n, t_1, \cdots, t_N) \cdot (x, \theta, z_1, \cdots, z_N) = (x, \theta + \sum_{i=1}^n s_i e_i + \sum_{i=1}^N t_i \eta_i, e^{\sqrt{-1}t_1} z_1, \cdots, e^{\sqrt{-1}t_N} z_N)$$

where (x, θ) is restriction of action-angle coordinates to the co-sphere bundle $S^*(T^n) \cong S^{n-1} \times \mathbb{R}^n / 2\pi\mathbb{Z}^n \subset \mathbb{R}^n \times T^n$ of the torus T^n, and e_i is the standard basis of $\mathbb{R}^n$. Moreover the moment map of this action is

$$(5.2)\quad \Phi \oplus \mu : S^*(T^n) \times \mathbb{C}^N \to (\mathbb{R}^n \oplus \mathbb{R}^N)^*, (x, \theta, z_1, \cdots, z_N) \mapsto x \oplus (\langle x, \eta_1 \rangle + \|z_1\|^2 - \kappa_1, \cdots, \langle x, \eta_N \rangle + \|z_N\|^2 - \kappa_N).$$

Now we consider contact quotient $M_\Delta := \mu^{-1}(0)/T^N$, then there are an induced contact form α_Δ, an induced T^n-action and an induced moment map Φ_Δ on M_Δ. Moreover the moment cone $C(\Psi_\Delta)$ coincides with $\Delta \cup \{0\}$ where Ψ_Δ is an universal moment map of $(M_\Delta, \alpha_\Delta, T^n, \Phi_\Delta)$.

Remark 5.17. *We can take a contact version of action-angle coordinates on the open dense subset $\Phi_\Delta^{-1}(\mathring{\Delta} \cap S^{n-1})$ since contact structures that on $\Phi_\Delta^{-1}(\mathring{\Delta} \cap S^{n-1})$ and $\Phi^{-1}(\mathring{\Delta} \cap S^{n-1})$ are contactomorphic. Then we can denote a contact form $\alpha_\Delta = \sum_{i=1}^n x_i d\theta_i$ and a Reeb vector field $\xi_\Delta = \sum_{i=1}^n x_i \frac{\partial}{\partial \theta_i}$.*

Remark 5.18. *One can easily see the compactness and connectedness of M_Δ in the above theorem by observing the pre-image of the moment map μ and Φ_Δ.*

Definition 5.19. *Let (M, D) be a contact manifold. We say that a torus $T \subset \mathfrak{Con}(M, D)$ is of* **Reeb type** *if there is a contact 1-form α such that its Reeb vector field lies in the Lie algebra $\mathfrak{t}$ of T.*

5.4. **Strongly convex cones and K-contact type manifolds.**

Lemma 5.20. *Let (M, D) be a contact manifold and take a torus $T \subset \mathfrak{Con}(M, D)$. Then the action of the torus T is of Reeb type if and only if there exist an $X \in \mathfrak{t}$ and an $\alpha \in \Gamma(D_+^\circ)$ such that $\langle \Phi_\alpha, X \rangle > 0$.*

Proof. If the torus action is of Reeb type, then we can take a contact form $\alpha' \in \Gamma(D_+^\circ)$ such that it Reeb vector field ξ' is contained $\mathfrak{t}$. Since α and $\alpha' \in \Gamma(D_+^\circ)$, we can take a positive function f such that $\alpha = f \cdot \alpha'$. Therefore $\langle \phi_\alpha, \xi' \rangle = f \cdot \alpha'(\xi') = f > 0$. Hence only if part holds. If there exist $X \in \mathfrak{t}$ such that $\langle \Phi_\alpha, X \rangle > 0$, then we set $\alpha' := \frac{\alpha}{\alpha(X)} \in \Gamma(D_+^\circ)$. One easily sees that $\alpha'(X) = 1$ and $\iota_X d\alpha' = 1$ holds. Therefore torus T is of Reeb type. □

Corollary 5.21. *Let (M, D) and T be as above. Let Ψ be a universal moment map of (M, D, T). Then the action of the torus T on M is of Reeb type if and only if there exists $X \in \mathfrak{t}$ such that $\langle \Psi, X \rangle > 0$.*

Lemma 5.22. *Let (M, D, T, Ψ) be a c.c.c.t. manifold. Then the action of the torus T on M is of Reeb type if and only if its moment cone $C(\Psi)$ is a strongly convex cone.*

Proof. If $C(\Psi)$ is weakly convex, then there exists $X \in \mathfrak{t}$ and $x \in \mathrm{Im}\Psi \setminus \{0\}$ such that $\langle x, X \rangle = 0$. Hence we can not take $X \in \mathfrak{t}$ such that $\langle \Psi, X \rangle > 0$. Therefore torus T is of non-Reeb type. Suppose $C(\Psi)$ is strongly convex. Note that we can denote $C(\Psi) = \{x \in \mathfrak{t}^* | \langle x, \eta_i \rangle \geqq 0, i = 1, \cdots, N\}$. Now take $X \in \sum_i^N a_i \eta_i$ for positive numbers $a_1, \cdots, a_N$. Then since $C(\Psi)$ is strongly convex, $\langle x, X \rangle = \sum a_i \langle x, \eta_i \rangle > 0$ for any $x \in \mathrm{Im}\Psi$. The result now follows from Corollary 5.21. □

Corollary 5.23. *Let (M, D, T, Ψ) be as above. There is a subtorus $K \subset T$ of Reeb type if and only if $C(\Psi)$ is a strongly convex.*

Proof. The moment map of K-action is given by $i^* \circ \Psi$ where $i : K \to T$ is inclusion. Since i^* is a projection from $\mathfrak{t}$ to $\mathfrak{k}$, we get the result. □

Lemma 5.24. *Let (M, D) be a $(2n-1)$-dimensional contact manifold. Then the dimension of a maximal torus in $\mathfrak{Con}(M, D)$ is at most n where maximality is given by inclusion property.*

Proof. Consider the lift of an action of $T \subset \mathfrak{Con}(M, D)$ on M to the T-action on the symplectization $(D_+^\circ, d(e^t\alpha))$ where α is some contact form and t is the radial coordinate. Since the symplectic form $d(e^t\alpha)$ is exact, T is a subset of $\mathrm{Ham}(D_+^\circ, d(e^t\alpha))$. The dimension of the torus acting in an effective Hamiltonian way on symplectic manifold X^{2n} is at most n (see [C], Theorem 27.3). Therefore $\dim T \leqq n$ holds. □

Proposition 5.25. *Let $(M, \alpha, \xi, \Phi, g, T, \psi)$ be a compact connected toric K-contact manifold with contact form α, a CR structure Φ and a metric g. Then $C(\Psi)$ is a strongly convex.*

Proof. First of all, note that we can assume without loss of generality α and g are T-invariant by averaging them over T. Hence $(M, \alpha, \xi, \Phi, g)$ is K-contact if and only if $\mathcal{L}_\xi g = 0$, that is, a Reeb flow F is contained by $\mathrm{Isom}_0(M, g)$ where $\mathrm{Isom}_0(M, g)$ is the identity component of the isometry group. Therefore F is a subgroup of the compact finite dimensional group $G := \mathrm{Isom}_0(M, g) \cap \mathfrak{Con}_0(M, \alpha)$. Moreover $T \subset G$ holds. Thus F is contained in the centralizer of T since F is in the center of G as follows form Lemma (5.12). On the other hand, Lemma (5.24) yields that T is maximal and in fact that the centralizer of a maximal torus is coincide to itself. Therefore $F \subset T$ holds, that is, $\mathrm{Lie}(F) \subset \mathfrak{t}$. Now consider the closure $\overline{F} :=$ the closure of F, then this is a subtorus in T of Reeb type. The result now follows from Corollary 5.23. □

Proof of Theorem 5.2 By Proposition 5.25 we have only to prove its converse. If the moment cone of (M, D, T, Ψ) is strongly convex, then we get a toric Sasakian structure on (M, D, T, Ψ) by Delzant construction as [L3]. Hence (M, D) is of Sasakian type, in particular of K-contact type. This completes the proof of Theorem 5.2.
As a result, strongly convex good cones correspond to of K-contact type, in particular of Sasakian type and weakly convex cones correspond to of non-Sasakian type contact manifold.

Remark 5.26. *In toric cases, it is true that K-contact type implies Sasakian type. since Theorem 5.2 holds, and we can construct a Sasakian structure which is induced by Delzant construction, but it is not true in generic cases.*

There is a well known property about the relation between K-contact type and Sasakian type as below:

Proposition 5.27 ([L4], Proposition 3.1)**.** *A compact contact manifold $(M, D = \ker\alpha)$ admits the structure of a K-contact manifold if and only if there exists a torus $T \hookrightarrow \mathfrak{Con}(M, \alpha)$ such that the T-action is of Reeb type.*

From this property and Theorem 5.2, we conclude that for any contact form, the group of strict contactomorphisms on a c.c.c.t. manifold of non-Sasakian type does not include a torus whose action is of Reeb type, in particular the intersection of $\mathfrak{Con}(M, \alpha)$ and Reeb flow F is $\{\mathrm{id}\}$.

Moreover the symplectic cones that correspond to a weakly convex cone do not have a Kähler cone structure in view of Remark 5.8, but they have the canonical Kähler structure that is determined by the cutting construction and hence c.c.c.t.manifolds of non-Sasakian type have the canonical almost contact metric structure as follows:

Let (M, D, T, Ψ) be a c.c.c.t. manifold with the canonical contact form α and suppose that a vector field ξ is its Reeb vector field. Now M is embedded in its symplectic cone $(D_+^\circ, d(e^t\alpha), T, \Psi)$ with the canonical Kähler structure $(J, d(e^t\alpha), h)$ as the pre-image of the intersection of the unit sphere and a moment cone under the moment map Ψ. Then we get an almost contact metric structure (Φ, ξ, α, g)satisfying

$$J\iota_* X = \iota_* \Phi X + \alpha(X)\xi,\ g := \iota^* h$$

where $\iota : M \to D_+^\circ$ is the inclusion map and $X \in \Gamma(TM)$.

Remark 5.28. *Generally speaking, every C^∞ orientable hypersurface of an almost complex manifold has an almost contact structure and if its ambient space is an almost Hermitian manifold then it has an almost contact metric structure (see* [T] *).*

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